NO between 2000–3000°K," J. Chem. Phys. 27, 850–857 (1957).

[2] Skinner, G. B. and Ruehrwein, R. A., "Shock tube studies on the pyrolysis and oxidation of methane," J. Phys. Chem. 63, 1736–1742 (1959).

[3] Palmer, H. B. and Knox, B. E., "Contact surface tailoring in a chemical shock tube," ARS J. 31, 826–828 (1961).

[4] Napier, D. H. and Simonson, J. R., "Attainment of interface matching in a chemical shock tube," ARS J. 32, 1736–1737 (1962).

[5] Knox, B. E., "Thermal decomposition of hydrazine in a single-pulse shock tube," Ph.D. Thesis, Pennsylvania State Univ. (1963).

[6] Resler, E. L., Jr., private communication, Dept. Aeronaut. Eng., Cornell Univ., Ithaca, N. Y. (October 1962).

Plane Poiseuille Flow of a Rarefied Gas

YOUNG-PING PAO* AND D. ROGER WILLIS†
Princeton University, Princeton, N. J.

CONSIDER the problem of rarefied gas flow through a long two-dimensional channel, of length L and width d, connecting two large reservoirs at slightly different pressures P_1 and P_2; the channel plates and the reservoirs are being kept at the same temperature θ.

Experiments measuring the flow rate through approximately two-dimensional channels with various mean pressures were made by a number of investigators, e.g., see Rasmussen.[1] The normalized volumetric flow rate, i.e., the mass flow rate multiplied by $R\theta$ and divided by $(P_1 - P_2)$, is denoted by T. Rasmussen observed a minimum in T as a function of the mean channel pressure, which is inversely proportional to the Knudsen number.

The existence of this minimum was demonstrated theoretically in a paper by Cercignani,[2] who analyzed the problem for an arbitrary Knudsen number by assuming a strictly linear pressure gradient along the channel and using a simple kinetic model to describe the intermolecular collisions.

In what follows, it will be shown that the knowledge of the flow behavior in two extreme limits, i.e., the Knudsen limit and the continuum limit, will be sufficient to establish the existence of this minimum.

Choose the coordinates system such that x is in the direction of the flow, and the plates are given by $0 \leq x \leq L$ and $y = \pm d/2$.

The Knudsen limit and the continuum limit are characterized by $L/\lambda \rightarrow 0$ ($d/\lambda \rightarrow 0$ a fortiori) and $\lambda/d \rightarrow 0$, respectively, where λ is the mean free path of the gas.

In the Knudsen limit, because of the absence of intermolecular collisions, it is possible and convenient to consider only the case in which $P_2 = 0$. This problem reduces to an integral equation for $n(x)$, the number of molecules striking a unit area of a channel plate about the point x in unit time, namely (see Ref. 3, part 1),

$$n(x) = \frac{P_1}{2m(2\pi RT)^{1/2}}\left[1 - \frac{x}{(x^2 + d^2)^{1/2}}\right] + \frac{d^2}{2}\int_0^L \frac{n(\zeta)d\zeta}{[(x-\zeta)^2 + d^2]^{3/2}} \qquad (1)$$

where R is the gas constant and m the mass of a single molecule. Equation (1) is referred to as the Clausing equation.

Received January 25, 1963. This research was supported by the U. S. Office of Scientific Research under Contract AF-AFOSR-142-63.

* Assistant in Research, Department of Aeronautical Engineering, Gas Dynamics Laboratory.

† Research Engineer, Department of Aeronautical Engineering, Gas Dynamics Laboratory.

The number of molecules which would escape into the vacuum end of the channel per unit cross-sectional area is given by the formula

$$Q_N = \frac{2}{Ld}\int_0^L n(x)[(x^2 + d^2)^{1/2} - x]dx \qquad (2)$$

For the derivation of Eqs. (1) and (2), refer to DeMarcus (Ref. 3, parts 1 and 3).

A complete solution to Eq. (1) has not been obtained. However, using a "squeezing" method developed by DeMarcus, the following bounds for $n(x)$ and Q_N have been found:[4]

$$\frac{P_1}{m(2\pi R\theta)^{1/2}}\left[1 - \frac{x}{L}\right] \geq n(x) \geq \frac{P_1}{m(2\pi R\theta)^{1/2}} \times \left[1 + \frac{4}{3(L+d)}\left(\frac{L}{2} - x\right)\right] \qquad (3)$$

for $0 \leqslant x \leqslant L/2$, and

$$\frac{P_1}{m(\pi R\theta)^{1/2}}\,\frac{d}{L}\log\frac{L + (L^2 + d^2)^{1/2}}{d} > Q_N > \frac{P_1}{m(\pi R\theta)^{1/2}} \times \frac{5}{6}\,\frac{d}{L}\log\frac{L}{d} \qquad (4)$$

A sufficient requirement for Eqs. (3) and (4) to be valid is that $L/d > 100$.

Return to the original problem where $P_2 \neq 0$, and denote by T the quantity $m \cdot Q_N \cdot R\theta/(P_1 - P_2)$, i.e., the normalized volumetric flow rate per unit cross-sectional area of the channel.

By virtue of the inequalities (4), the bounds for T are

$$\left(\frac{R\theta}{2\pi}\right)^{1/2}\frac{5}{6}\,\frac{d}{L}\log\frac{L}{d} < T_{Kn} < \left(\frac{R\theta}{2\pi}\right)^{1/2}\frac{d}{L} \times \log\left(\frac{L + (L^2 + d^2)^{1/2}}{d}\right) \qquad (5)$$

where subscript Kn indicates the value for Knudsen flow. It may be worthwhile to stress that the bounds given in (5) are exact when $L/d > 100$. For the derivation of inequalities (3–5), see Ref. 4.

On the other hand, for continuum conditions, the flow rate is known to obey the classical Poiseuille formula, namely,

$$T = \frac{d^2 P_m}{12\mu L} = \frac{d}{L}\,\frac{d}{\lambda}\,\frac{(\pi R\theta)^{1/2}}{12(2)^{1/2}} \qquad (6)$$

where P_m is the mean pressure, and the viscosity μ is given by $\mu = \frac{1}{2}\rho\bar{c}\lambda$, where ρ, $\bar{c}$, and λ are the density, the mean thermal speed, and the mean free path of the gas, respectively. Equation (6) holds, providing that the channel is long enough that the end effects become negligible.

Formula (6) will be assumed to be valid for $d/\lambda > 100$. It can be seen by comparing Eq. (6) with the inequality (5) that

$$T(d/\lambda = 100) < T_{Kn}$$

provided $L/d > 2700$.

Therefore, with a channel having $L/d > 2700$ if one starts in the Knudsen flow regime and increases the ratio d/λ, T initially varies from T_{Kn} to a lower value $T(d/\lambda = 100)$ and then increases without bound according to formula (6). It thus becomes apparent that, for a sufficiently long channel, T must assume a minimum in the region between the Knudsen flow and the continuum flow. In other words, this implies the existence of a minimum in T as a function of the ratio d/λ.

References

[1] Rasmussen, R. E. H., "Über die Strömungen von Gasen in engen Kanalen," Ann. Physik **5**, 29 (1937).

[2] Cercignani, C., "Plane Poiseuille flow and Knudsen minimum effect," 3rd International Symposium on Rarefied Gas Dynamics, Paris (June 1962).

[3] DeMarcus, W. C., "The problem of Knudsen flow," U. S. Atomic Energy Commission Rept. K-1302, ORGDP (1956).

[4] Pao, Y. P., "Some results for plane Poiseuille flow of rarefied gases," Rept. 619, Dept. Aeronaut. Eng., Princeton Univ. (1962).

Nonequilibrium Plasma Characteristics in Hypersonic One-Dimensional Flow

FRED MEYER,* W. K. H. KRESSNER,† AND J. R. STEFFEY‡

Boeing Company, Seattle, Wash.

FOR the case of one-dimensional steady flow in a streamtube of constant cross section with negligible viscosity and diffusion, the kinetic and gasdynamical flow equations have been combined in a single set of differential equations describing the composition and flow of a system of reacting gases. A rather complete chemical model considering 10 reversible reactions is used together with reaction rates published by Teare, Kivel, Hammerling, et al.[1] Investigation of the chemical rate equation with a given set of rate constants leads to the determination of temperature, air-density, and particle-concentration histories behind the shock front.

The following set of equations comprises the heart of the program:

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{v}\frac{dv}{dx} = 0 \qquad (1)$$

$$\frac{\rho v}{p}\frac{dv}{dx} + \frac{1}{p}\frac{dp}{dx} = 0 \qquad (2)$$

$$v\frac{dv}{dx} + \frac{dh}{dx} = 0 \qquad (3)$$

$$\frac{1}{\rho}\frac{d\rho}{dx} - \frac{1}{p}\frac{dp}{dx} + \frac{1}{T}\frac{dT}{dx} - \frac{1}{\overline{W}}\frac{d\overline{W}}{dx} = 0 \qquad (4)$$

$$\frac{1}{\overline{W}}\frac{d\overline{W}}{dx} = -\frac{\overline{W}\Sigma_i\gamma_i}{\rho v} \qquad (5)$$

$$\frac{dh}{dx} - \bar{c}_p\frac{dT}{dx} = \frac{\Sigma_i H_i\gamma_i}{\rho v} \qquad (6)$$

The first three equations are the differential forms of the overall equations of continuity, conservation of momentum, and conservation of energy, respectively. Equation (4) is the differential form of the gas law or thermal equation of state, where

$$\overline{W} = \Sigma_i(Y_i/W_i)^{-1} = \Sigma_i W_i r_i/\Sigma_i r_i$$

is the average molecular weight of the mixture of the ith species, W_i is the gram molecular weight of the ith species, $r_i = \rho(Y_i/W_i)$ is the concentration in moles per unit volume

Received January 14, 1963.

* Research Specialist, Aerospace Division.

† Research Engineer, Aerospace Division.

‡ Associate Engineer, Aerospace Division.

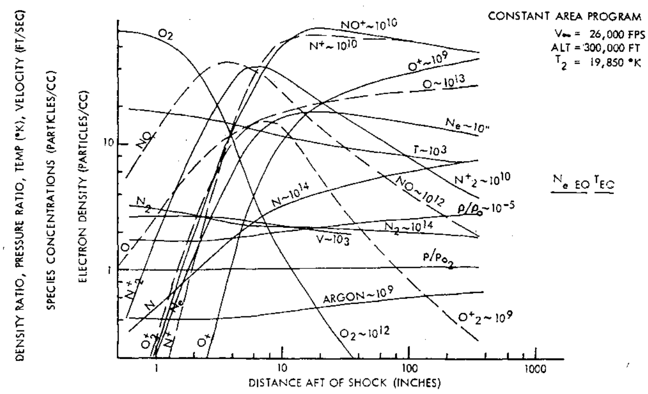

Fig. 1 Species and flow-property histories aft of normal shock

of the ith species, and Y_i/W_i is the number of moles of species i per gram of original air.

The fifth equation is the differential form of $\overline{W} = \Sigma_i(Y_i/W_i)^{-1}$ after application of the equations of continuity for the individual species in the flowing mixture, neglecting diffusion:

$$\rho v(dY_i/dx) = W_i\gamma_i$$

where γ_i is the rate at which species i is produced per unit time-per unit volume and is computed from the postulated chemical mechanism according to the standard methods of chemical kinetics.

The last equation is the differential form of the enthalpy of the mixture of gases, where $\bar{c}_p = \Sigma_i Y_i c_{pi}$ is the average heat capacity at constant pressure, and H_i is the molar heat capacity of the ith species. This assumes that $[(\partial h_i/\partial p)]_T = 0$, a good approximation for the high temperatures and low pressures corresponding to re-entry conditions. The thermodynamic functions required for each species were computed from its partition function using the approximations developed by Hochstim for these species.[2]

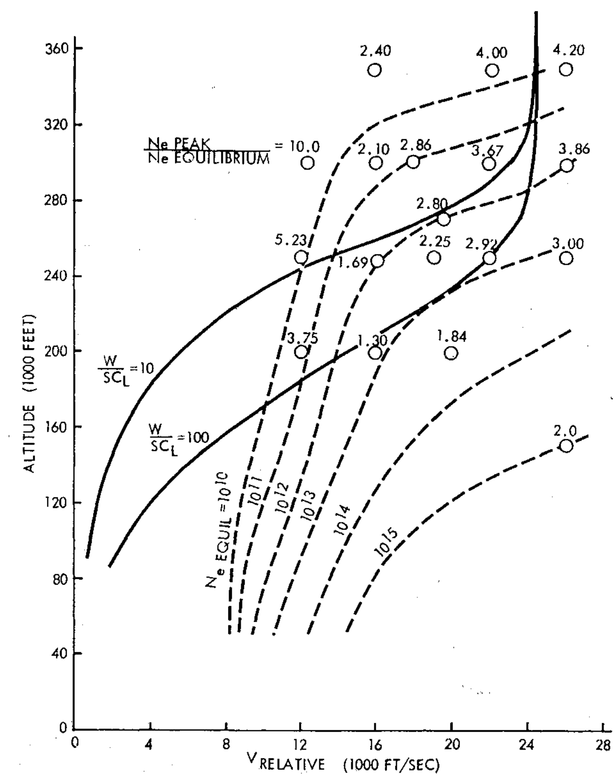

Fig. 2 Peak electron-density overshoots aft of normal shock